J478. Prove that in any triangle ABC the following inequality holds:

$$4\left({\ell_a}^2 + {\ell_b}^2 + {\ell_c}^2\right) \le (a+b+c)^2.$$

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We have that

$$l_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}} = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \le \sqrt{s(s-a)}.$$

Therefore,

$$l_a \le \sqrt{s(s-a)}$$

and similarly $l_b \le \sqrt{s(s-b)}$ and $l_c \le \sqrt{s(s-c)}$.

Hence,

$$4(l_a^2 + l_b^2 + l_c^2) \le 4s(3s-a-b-c) = 4s^2 = (2s)^2 = (a+b+c)^2.$$

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